 ISBN : 978-0-9742114-1-9

An alternative marginal utility growth proxy for use in asset pricing models

*David J. Moore**

*The University of Memphis, Fogelman College of Business and Economics. Email address: djmphd@gmail.com. Earlier versions of this paper were presented at Pepperdine University, The University of Memphis, The University of Tennessee, Lynchburg College, and during the 2009 Ph.D. Project Finance Doctoral Student Association meetings coincident with the 2009 Western Finance Association annual meeting. The author would like to thank participants of those presentations for their constructive comments and those who submitted comments through SSRN. An earlier version of this paper was published in the Financial Study Association Rotterdam (FSR) Forum, September 2009, Issue 5

 ISBN : 978-0-9742114-1-9

1 Introduction

The exchange economy model of Lucas yields the familiar Euler condition that discounts expected returns by marginal utility growth. Unfortunately, proxies for marginal utility growth are subject to theoretical and empirical shortcomings. This study proposes a new macroeconomic theory-derived and productivity-based marginal utility growth proxy that avoids the theoretical difficulty in utility function specification and empirical problems associated with consumption data used as the primary utility function input. The theoretical and empirical characteristics of this proxy indicate it may be useful in linear asset pricing models as an instrumental variable.

2 Exchange economy model

The exchange economy model (Lucas, 1978) yields the Euler condition for a representative agent's intertemporal utility maximization:

$$p_t u' [c_t] = \beta E_t \left[u' [c_{t+1}] (p_{t+1} + d_{t+1}) \right] \quad (1)$$

where E_t is the time t expectations operator, p_t is the price at time t, c_t is the consumption at time t, β is the rate of time preference (subjective discount factor), and d_{t+1} are dividends paid at time $t+1$. The equality in (1) reveals the cost in marginal utility ($p_t u' [c_t]$) of purchasing the asset must be equal to the discounted (β) expected gain of the future payoff ($u' [c_{t+1}] (p_{t+1} + d_{t+1})$). By dividing both sides by $p_t u' [c_t]$ and defining $R_{t+1} \equiv (p_{t+1} + d_{t+1}) / p_t$ equation (1) can be expressed in discount factor form:

$$E_t [m_{t+1} R_{t+1}] = 1 \quad (2)$$

where

$$m_{t+1} = \beta \frac{u' [c_{t+1}]}{u' [c_t]} \quad (3)$$

The stochastic discount factor (3) is unobservable therefore explicit utility function assumptions must be made before applying the model. Two frequently used utility functions, constant relative risk aversion (CRRA) utility and log utility are now used to illustrate theoretical and empirical difficulties associated with marginal utility growth proxies.

CRRA utility CRRA utility can be expressed in the power-utility form $u[c] = c^{1-\theta}/(1-\theta)$ where θ, a constant, is the relative risk aversion coefficient.

 ISBN : 978-0-9742114-1-9

Taking the first derivative and inserting into (3):

$$m_{t+1} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\theta} \tag{4}$$

This discount factor is at the heart of the "equity premium puzzle" identified by Mehra and Prescott (1985). The puzzle can be illustrated by the lower bound on the discount factor coefficient of variation $CV[m] = \sigma[m]/E[m]$ provided by Hansen and Jagannathan (1991):

$$\frac{|E[R^e]|}{\sigma[R^e]} \leq \frac{\sigma[m]}{E[m]} \tag{5}$$

where R^e is the return in excess of the Treasury Bill rate. The post-war Sharpe ratio $(E[R^e]/\sigma[R^e])$ for the value-weighted NYSE portfolio is about 0.5 while per-capita consumption growth has a standard deviation of about 1% per year (Cochrane, 2005). Given the discount factor of equation (4), and setting $\beta = 1$, the required coefficient of risk aversion θ to satisfy (5) is approximately 50. However, Mehra (2003) noted a large body of literature that suggests the coefficient of risk aversion θ is "certainly less than 10." The extremely high risk aversion indicates the volatility of consumption growth is too low to explain the equity premium.

CRRA utility does not fit observed data without implausibly high levels of risk aversion, but what about other utility function forms? Other more complex utility functions have been proposed such as habit formation and time-non-separable forms. Unfortunately, the more complex forms have proven unsuccessful in resolving the equity premium puzzle as well (Mehra, 2003).

From log utility to CAPM Log utility $u[c_t] = \ln[c_t]$ can be shown to map directly into CAPM. Let R_t^W represent the return to the market wealth portfolio with the S&P 500 return often used as an empirical proxy. Cochrane (2005) found that given log utility,

$$R_{t+1}^W = \frac{1}{\beta} \frac{u'[c_t]}{u'[c_{t+1}]} \tag{6}$$

Since $m_{t+1} = \beta u'[c_{t+1}]/u'[c_t]$, $R_{t+1}^W = 1/m_{t+1}$ or $m_{t+1} = 1/R_{t+1}^W$. Taking the linear approximation we arrive at the CAPM discount factor:

$$m_{t+1} \approx a_0 + a_1 R_{t+1}^W \qquad a_1 < 0 \tag{7}$$

 ISBN : 978-0-9742114-1-9

Given the relatively high volatility of R^W, the stochastic discount factor (7) has an inherent advantage over the low volatility consumption growth-derived discount factor (4). However, the inability of CAPM to explain anomalies such as the size premium, the value premium, and momentum profits is indicative of model mis-specification. Cochrane (2005) noted that if the discount factor (7) is to price multiple assets simultaneously, the coefficients can *not* be constant and the correct discount factor representation is

$$m^*_{t+1} = a_{0t} + a_{1t} R^W_{t+1} \tag{8}$$

where a_{0t} and a_{1t} are time-varying coefficients. Cochrane also noted these time-varying coefficients account for time-varying expected returns, variances, covariances, and risk free rates. Here, I provide an additional perspective: the time varying coefficients of (8) also account for time-varying risk aversion, specifically fluctuations about $\theta = 1$ given the connection between log-utility and CAPM.

3 Macroeconomic growth model

Since the time-varying coefficients in (8) ultimately capture time-varying marginal utility growth, I pursue an alternative expression for marginal utility growth used to drive the time variability of a_{0t} and a_{1t} using a discrete time general equilibrium macroeconomic growth model.

The model The model developed here follows the closed economy exogenous growth model without government of King and Rebelo (1999) and is illustrated in Figure 1.

[Figure 1 about here.]

To ensure the existence of a steady-state, exogenous growth is introduced via labor augmentation consistent with Sala-i Martin (1990). As such, the production function can be specified in Cobb-Douglas form as

$$Y_t = A_t F\left[K_t, N_t X_t\right] = A_t K_t^{1-\alpha} \left(N_t X_t\right)^{\alpha} \tag{9}$$

where Y_t is GDP, K_t is capital input, N_t is labor input, A_t is the random productivity shock, and X_t is the deterministic component of productivity which grows at a constant (and exogenous) rate $\gamma > 1$:

$$X_{t+1} = \gamma X_t \tag{10}$$

 ISBN : 978-0-9742114-1-9

I am not addressing labor-leisure choice in this study; therefore, labor is assumed to be fixed ($N_t = 1 \ \forall t$) and the production function simplifies to:

$$Y_t = A_t K_t^{1-\alpha} X_t^{\alpha} \tag{11}$$

The Cobb-Douglas production function was chosen for several reasons. First, by construction this production function exhibits constant returns to scale, consistent with the empirical findings of Jorgenson (1972). Second, the empirical evidence noted by Jorgenson (1972) suggests the estimated elasticity of substitution for the constant elasticity of substitution (CES) production function is not significantly different from unity and therefore the CES reduces to Cobb-Douglas form. Third, Arroyo (1996) suggests the Cobb-Douglas form is "probably more descriptive of aggregate technological conditions."

Maximization problem Following the model of King and Rebelo (1999), all trending variables are scaled by X_t and transformed variables are denoted by lower case letters (e.g., $y_t = Y_t/X_t$). The infinitely-lived central planner maximizes discounted expected utility:

$$\max_{c_t, k_{t+1}} \left\{ E_0 \left[\sum_{t=0}^{\infty} b^t u\left[c_t\right] \right] \right\} \tag{12}$$

where $b^t < 0$ is the rate of time preference, subject to several constraints. First, all output is either consumed or invested:

$$y_t = c_t + i_t \tag{13}$$

In addition, capital stock evolves according to the perpetual inventory method:

$$\gamma k_{t+1} = i_t + (1-\delta)k_t \tag{14}$$

where δ is the rate of depreciation. Noting $y_t = A_t k_t^{1-\alpha}$ and combining constraints (13) and (14):

$$\gamma k_{t+1} = A_t k_{t+1}^{1-\alpha} - c_t + (1-\delta)k_t \tag{15}$$

the Lagrangian is:

$$\mathcal{L} = \sum_{t=0}^{\infty} b^t u\left[c_t\right] + \sum_{t=0}^{\infty} b^t \lambda_t \left(A_t k_t^{1-\alpha} + (1-\delta)k_t - c_t - \gamma k_{t+1}\right) \tag{16}$$

 ISBN : 978-0-9742114-1-9

Solution The first order conditions of the maximization problem are:

$$\frac{\partial \mathcal{L}}{\partial c_t} = b^t u'\left[c_t\right] - b^t \lambda_t = 0$$

$$u'\left[c_t\right] = \lambda_t \tag{17}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = b^{t+1}\lambda_{t+1}\left(A_{t+1}(1-\alpha)k_{t+1}^{-\alpha} + (1-\delta)\right) - b^t\lambda_t\gamma = 0$$

$$\frac{\lambda_{t+1}}{\lambda_t} = \frac{\gamma}{b\left(A_{t+1}(1-\alpha)k_{t+1}^{-\alpha} + (1-\delta)\right)} \tag{18}$$

The first order conditions (17) and (18) can be combined to reveal the key contribution of this study: an alternative proxy for marginal utility growth:

$$\Gamma_{t+1} = \frac{\lambda_{t+1}}{\lambda_t} = \frac{u'\left[c_{t+1}\right]}{u'\left[c_t\right]} = \frac{\gamma}{b\left(A_{t+1}(1-\alpha)k_{t+1}^{-\alpha} + (1-\delta)\right)} \tag{19}$$

This expression provides a readily observable proxy for unobservable marginal utility growth that avoids the theoretically troublesome utility function specification and empirically troublesome consumption data. Now I proceed towards connecting Γ with the time-varying discount factor coefficients a_{0t} and a_{1t}.

4 Empirical construction and characteristics of Γ

Table 1 summarizes the sources, frequency, and availability of the data used to construct Γ. Aggregate macroeconomic data are obtained from the Bureau of Economic Analysis National Income and Product Accounts tables (BEA NIPA tables). Using the BEA NIPA aggregate macroeconomic data, data, output (Y) is defined as gross domestic product (GDP), capital (K) is defined as private non-residential fixed assets in place, investment (I) is defined as private non-residential fixed investment, and consumption (C) is defined as non-durable consumption. All nominal data are converted to real using the CPI deflator from the U.S. Bureau of Labor Statistics (BLS).

[Table 1 about here.]

Unfortunately quarterly capital data are unavailable from the Bureau of Economic Analysis therefore the series must be estimated. The quarterly capital series is constructed from annual capital data and quarterly investment data following the procedure of Balvers and Huang (2007). Capital in

 ISBN : 978-0-9742114-1-9

quarter q is computed as:

$$K_{y,q} = \left(\frac{\sum_{i=1}^{q} I_{y,i}}{\sum_{i=1}^{4} I_{y,i}} \right) (K_y - K_{y-1}) + K_{y-1} \tag{20}$$

with $K_{y,q}$ equal to capital for quarter q in year y, $I_{y,i}$ equal to investment for quarter i in year y, and K_y equal to capital at the end of year y.

Following the parametrization of King and Rebelo (1999), the value for the rate of time preference b is set to 0.984, the labor share of output α is set to $2/3$, the quarterly growth rate γ is set to 1.004, and the quarterly depreciation rate δ is set to 0.025. Given this parametrization and current dataset, $E[\Gamma] = 1.0140$, $\sigma[\Gamma] = 0.0030$, and $\sigma[\Gamma]/E[\Gamma] = 0.0029$.

5 [Un]conditional estimation using Γ

The low volatility and coefficient of variation of Γ render it unsuitable for use as a stand-alone discount factor since it will not satisfy the Hansen-Jagannathan bounds. However, the low volatility may be beneficial in a conditional estimation from two perspectives. First, from a *scaled payoff* perspective, a low volatility instrument is representative of a feasible trading strategy due to low transactional activity and costs. Second, from a *scaled discount factor* perspective, when the coefficients of discount factor (8) are allowed to vary with Γ, they capture changes in risk aversion around unity.

If we allow a_{0t} and a_{1t} to be functions of Γ_t, the conditioning information in the original Euler condition can be stated explicitly:

$$E[m_{t+1}R_{t+1}|\,\Gamma_t] = 1 \tag{21}$$

Of course, whenever one discusses instruments they are exposed to the criticism of omitting instruments relevant to the estimation. However, given the equality between the instrument and marginal utility growth, I proceed under the assumption that the relevant information is captured by Γ.

Cochrane (2005) noted that an unconditional Euler condition is implied when using a discount factor with constant coefficients, as in equation (7). Thus, if there were some way to express the discount factor with time-varying coefficients, equation (8), in a form with constant coefficients, an unconditional Euler condition could also be used.

To do so, begin by defining a_{0t} and a_{1t} as linear functions of Γ_t:

$$a_{0t} = a_{00} + a_{01}\Gamma_t \tag{22}$$
$$a_{1t} = a_{10} + a_{11}\Gamma_t \tag{23}$$

 ISBN : 978-0-9742114-1-9

Inserting these coefficients into (8) yields:

$$m_{t+1}^{*} = a_{00} + a_{01}\Gamma_t - a_{10}R_{t+1}^{W} - a_{11}\left(\Gamma_t R_{t+1}^{W}\right) \tag{24}$$

In other words, a conditional estimation (conditioned on Γ_t) with a single factor $\left(R_{t+1}^{W}\right)$ is equivalent to an *unconditional* estimation with three factors $\left(\Gamma_t,\ R_{t+1}^{W},\ \text{and } \Gamma_t R_{t+1}^{W}\right)$:

$$E\left[m_{t+1}R_{t+1}|\,\Gamma_t\right] = 1 \ \Rightarrow\ E\left[m_{t+1}^{*}R_{t+1}\right] = 1 \tag{25}$$

where m_{t+1} is represented by equation (7) and m_{t+1}^{*} is represented by equation (24).

6 Empirical evidence of refined discount factor efficacy

Here I show that the coefficient of variation (CV) of the refined discount factor with time-varying coefficients $\left(m_{t+1}^{*}\right)$ is greater than that of the original discount factor with fixed coefficients (m_{t+1}). The larger CV suggests a greater likelihood of satisfying the Hansen-Jagannathan (H-J) bounds.

It is quite common in financial literature to examine excess returns: $R_t^e = R_t - R_t^j$ where R^j represents any arbitrary return (typically the risk free rate). The Euler condition for excess returns is $E_t\left[m_{t+1}R_{t+1}^{e}\right] = 0$. Note that in this form $E\left[(2m)\,R^e\right] = 0$ is also true. As such, the coefficients of m are not identified. Therefore some form of normalization must be chosen. Here I choose the $E\left[m\right] = E\left[m^{*}\right] = 1$ normalization. With this choice of normalization, equations (7) and (24) are:

$$m = 1 + a_1\tilde{f}_1 \tag{26}$$

$$m^{*} = 1 + a_{01}\tilde{f}_2 + a_{10}\tilde{f}_1 + a_{11}\tilde{f}_3 \tag{27}$$

where $\tilde{f}_1 = R_{t+1}^{W} - E\left[R_{t+1}^{W}\right]$, $\tilde{f}_2 = \Gamma_t - E\left[\Gamma_t\right]$, and $\tilde{f}_3 = \Gamma_t R_{t+1}^{W} - E\left[\Gamma_t R_{t+1}^{W}\right]$. Although the choice of normalization seems arbitrary, Cochrane (2005) noted that the choice is purely one of convenience and the pricing errors are independent of the choice of normalization.

The goal is to demonstrate the coefficient of variation for m^{*} is greater than that of m:

$$\frac{\text{var}\left[m^{*}\right]}{E\left[m^{*}\right]} > \frac{\text{var}\left[m\right]}{E\left[m\right]} \tag{28}$$

To demonstrate (28) the coefficients $(a_1, a_{01}, a_{10}, a_{11})$ must be obtained. To obtain a_1, I apply the Euler condition to an arbitrary excess return R_{t+1}^{e}:

$$0 = E\left[\left(1 - a_1\tilde{f}_1\right)R_{t+1}^{e}\right]$$

 ISBN : 978-0-9742114-1-9

Through some algebraic manipulation, it can be shown that:

$$a_1 = \frac{-E\left[R^e\right]}{\text{cov}\left[R^W, R^e\right]}$$

Using average quarterly return data extracted from the Kenneth E. French website, and arbitrarily choosing prior-return portfolio 4 ($M04$) excess returns, I find that $a_1 = -7.55$. After substituting the value of a_1 into (26) I find that $CV\left[m\right] = 35.15$.

Now, on to m^*. A bit more work must be done to obtain (a_{01}, a_{10}, a_{11}). Specifically, since there are three coefficients to obtain I must apply the Euler condition to three arbitrary excess returns R^{e1}, R^{e2}, R^{e3}. Again using average quarterly return data from the Kenneth E. French website, and arbitrarily choosing prior-return portfolios 4-6 ($M04, M05, M06$) I find that $a_{01} = -153.05$, $a_{10} = -140.57$, and $a_{11} = 165.11$. After substituting these coefficient values into (27) I find that $CV\left[m^*\right] = 40.47$. Therefore, at least with the arbitrarily chosen momentum portfolio excess returns, the coefficient of variation for the productivity-augmented discount factor is larger than that of the standard CAPM discount factor ($CV\left[m^*\right] > CV\left[m\right]$).

7 Conclusion

The macroeconomic growth model yields a productivity-based proxy for marginal utility growth that bypasses two key drawbacks of traditional marginal utility growth proxies. First, the difficulty in obtaining explicit utility functions, which are unobservable, is bypassed by using an observable productivity-based proxy. Second, the more reliably measured production data avoids the measurement error associated with consumption data.

A refined discount factor was constructed using the necessarily time-varying coefficients expressed as linear functions of the new marginal utility growth proxy. The new discount factor was shown to map directly into an unconditional estimation framework. In particular, the time-varying coefficients of the refined discount factor account for time-varying expected returns, variances, covariances, risk free rates, and risk aversion.

Some empirical evidence was presented to show the coefficient of variation of the refined discount factor is larger than that of the standard CAPM discount factor. As such, the refined discount factor with time-varying coefficients is more likely to satisfy the H-J bounds and may be able to price those assets that the standard CAPM discount factor with fixed coefficients can not.

References

Arroyo, Cristino R., 1996, Testing a production-based asset-pricing model, *Empirical Inquiry* pp. 357–377.

Balvers, Ronald J., and Dayong Huang, 2007, Productivity-based asset pricing: Theory and evidence, *Journal of Financial Economics* 86, 405–445.

Cochrane, John H., 2005, *Asset Pricing* (Princeton University Press).

Hansen, Lars P., and Ravi Jagannathan, 1991, Implications of security market data for models of dynamic economies, *Journal of Political Economy* 99, 225–262.

Jorgenson, Dale W., 1972, Investment behavior and the production function, *Bell Journal of Economics* 3, 220–251.

King, Robert G., and Sergio T. Rebelo, 1999, Resuscitating real business cycles, in John B. Taylor, and Michael Woodford, ed.: *Handbook of Macroeconomics* (Amsterdam: North Holland).

Lucas, Jr., Robert E., 1978, Asset prices in an exchange economy, *Econometrica* 46, 1426–14.

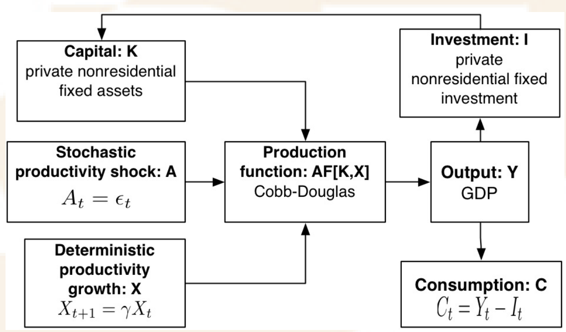
Mehra, Rajnish, 2003, The equity premium: Why is it a puzzle?, *Financial Analysts Journal* pp. 54–69.

———, and Edward Prescott, 1985, The equity premium: A puzzle, *Journal of Monetary Economics* 15, 145–161.

Sala-i Martin, Xavier, 1990, Lecture notes on economic growth, Working Paper No. 3563.

 ISBN : 978-0-9742114-1-9

Fig. 1: Graphical representation of general equilibrium macro growth model



 ISBN : 978-0-9742114-1-9

Tab. 1: Data items, sources, and availability

Monthly return data are converted to quarterly by compounding three single month returns. Monthly CPI data are converted to quarterly by taking three month averages. Annual capital data are converted to quarterly using equation (20). All BEA and CPI data are nominal not-seasonally adjusted data.

Item(s)	Source	Frequency	Availability
One-month T-Bill rate $\left(R^f\right)$	Kenneth French website	monthly	1927.01-2008.12
Market return proxy $\left(R^W\right)$	Kenneth French website	monthly	1927.01-2008.12
Momentum portfolio excess returns $(M01 - M10)$	Kenneth French website	monthly	1927.01-2008.12
aggregate labor (L)	BEA NIPA tables	quarterly	1947:1-2007:4
aggregate capital (K)	BEA NIPA tables	annual	1929-2006
aggregate private nonresidential investment (I)	BEA NIPA tables	quarterly	1947:1-2006:4
aggregate output (Y) and consumption (C)	BEA NIPA tables	quarterly	1947:1-2006:4
CPI deflator	BLS LABSTAT	monthly	1947.01-2007.12